MAP 499
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Relationship between structural behavior and component coordination in bistable lattices

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Introduction

Bistable lattices are topological models used to demonstrate the behavior of various materials. Our work uses **combinatorial** and **variational** approaches to assess the relationship between **geometrical constraints** and **minimal-energy states** of various bistable lattices including, but not limited to, **triangular** and **Penrose** lattices. This is useful in predicting the asymptotic response of an external energy source on a lattice structure given certain boundary conditions, rendering our results applicable to metamaterial design and understanding crystallization, among others.

Methods

Theoretical Overview

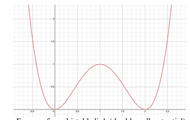
To calculate the total energy of a lattice, we examine the energy of each edge based on a model adapted from [1]: $h_c(r) = r^2(r-c)^2$,

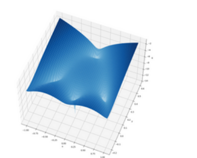
where r is the strain on the link between two neighboring node i and j, defined as: $s_{ij} = \frac{|(\mathbf{x}_i + \mathbf{u}_i) - (\mathbf{x}_j + \mathbf{u}_j)| - |\mathbf{x}_i - \mathbf{x}_j|}{|\mathbf{x}_i - \mathbf{x}_j|}$

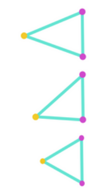
We calculate the total energy as $E(\mathbf{U}) = \sum_{(i,j)\in L} h(s_{ij})$,

with L being the set of all links) with respect to the vector of the displacement on each node U = [$u_1, u_2, \ldots, u_n$].

To examine minimal-energy states from a given displacement field, we turn the research into an **optimization problem**, minimizing the objective function E(U) with U as input. We also computed and implemented the **Jacobian** and the **Hessian** of the energy.


Energy of one bistable link (double well potential)


Energy of a triangular lattice with two fixed nodes and one free node

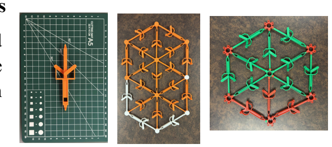


Computer Simulation

Implemented the gradient descent, Broyden-Fletcher-Goldfarb-Shanno (BFGS) optimization, Sequential Least Squares Quadratic Programming (SLSQP), and Dynamic Relaxation

3D Printed Structures

We used 3D printed models of bistable lattices to confirm our findings:

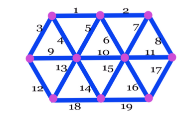


Analyzing the Conditions on Lattice Configurations

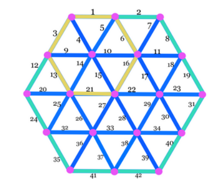
For each hexagonal subsection in a lattice, **the total elongation of the rims is equal to the total elongation of the spokes** [1]. Define a **compatible** lattice as one that follows these conditions, and define a ***still state*** as a compatible lattice configuration where the lattice has 0 energy (where its links are either in long or short mode). Using **counting arguments**, we:

- find the total possible still states in a simple hexagonal subsection of a triangular lattice
- find the total possible still states in a subsection of a triangular lattice with two overlapping hexagons
- determine whether compatibility and stillness on the micro-level implies the same on the macro-level


Example of a triangular lattice with one hexagon



Example of a triangular lattice with two overlapping hexagons


Example of a triangular lattice comprised of small, overlapping hexagons that make up a larger hexagon

Conclusion

Our research seeks to grow our understanding of the behavior of bistable lattices:

- We developed analytical and numerical methods to **simulate** and **predict** the resulting minimal-energy configurations of bistable lattice structures under external loading conditions.
- We found numerical evidence of the existence of **minimal but nonzero** energy states.
- We designed **3D-printable bistable spring mechanisms** with tunable energy barrier, and construct a real-life realization of the underlying mathematical problem.
- We **developed counting arguments** to quantify still states of a triangular lattice, made possible by its geometrical constraints.
- **Compatible, *still* lattices in the micro scale are necessarily compatible and still in the macro scale.**

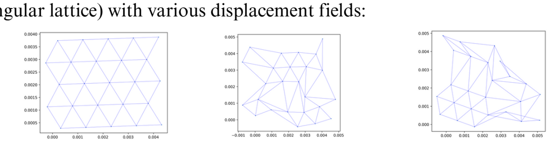
Future research directions:

- Investigate why minimal energy states may have nonzero energy
- Develop more generalized counting arguments that compute the total number of still states of an arbitrary triangular lattice, considering how connectivity and organization of the lattice impact the total number of still states.

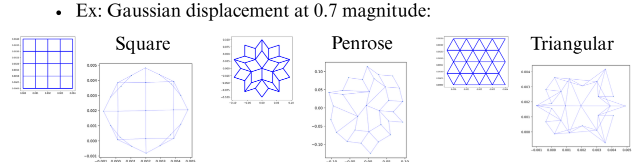
Results

Computer Simulations

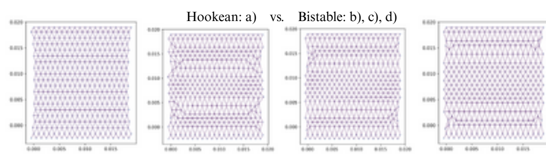
- There are 3 types of results from running the minimizer on a single lattice (ex: triangular lattice) with various displacement fields:



- Various lattices are examined under the same conditions for comparisons:
  - Ex: Gaussian displacement at 0.7 magnitude:



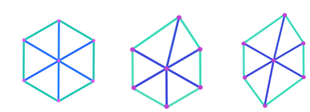
- Hookean (linear) lattices have unique solutions, bistable lattices have various distinct solutions:

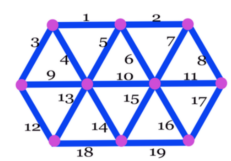


Analyzing Lattice Configuration Conditions

We found the total possible still states of a simple hexagonal subsection of a triangular lattice with a counting argument based on Cherkaev's conditions.. Adding these together gives us **924 possible still states** for a simple hexagonal subsection of a triangular lattice.

# of rims/spokes elongated	Total configurations
0	1
1	$({}^6C_1)^2 = 36$
2	$({}^6C_2)^2 = 225$
3	$({}^6C_3)^2 = 400$
4	$({}^6C_4)^2 = 225$
5	$({}^6C_5)^2 = 36$
6	1


Possible still configurations of a hexagonal subsection of a triangular lattice



Example of a triangular lattice with two overlapping hexagons

To find the total possible still states in a subsection of a triangular lattice with two overlapping hexagons, we assume that one of the hexagons has 924 possible still states and count how many configurations of the second hexagon are possible for each one. This gives us a total of **27,048 possible still states**.

By adding the rims and spokes of each small hexagon in a large lattice, we are able to determine that **a lattice that is still and compatible on the micro-level is still and compatible on the macro-level**.

References & Acknowledgements

[1] A. Cherkaev, A. Kouznetsov, and A. Panchenko. Still states of bistable lattices, compatibility, and phase transition. Springer Nature, 22:421–444, 2010.

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